B

Appendix B. Coordinate Transformation

B.1. Coordinate Transformation Relations

In this appendix we derive and discuss the transformation relations between curvilinear and orthogonal coordinate systems, and present an example using the Boussinesq equations. Much of what follows is taken from the book of Thompson *et al.*, *Numerical Grid Generation* (Thompson *et al.*, 1985; hereafter TWM), Sharman *et al.* (1988), Fletcher (1988), and Shyy and Vu (1991). Although the transformation relations are general and appear almost exclusively in two dimensional form in other references, we provide a complete set of relevant equations valid in three dimensions.

The use of a generalized coordinate implies that a distorted region in physical space will be mapped to a regular, rectangular region in computational space (Fletcher, 1988). Consequently, it is extremely convenient to perform all computations in the transformed space where the grid mesh is uniform and Cartesian. Consequently, techniques appropriate for standard Cartesian models can be applied directly without modification. This simplicity does not come without a price, however, as one must pay close attention to how the transformation metrics, which relate the physical grid to its computational counterpart, are discretized (*e.g.*, Thomas and Lombard, 1978). We will return to this subtle yet important point after developing the transformation relations and applying them to the governing dynamic equations.

Consider a general transformation from physical Cartesian coordinates (x,y,z,t) to curvilinear coordinates (ξ,η,ζ,t), both in right-handed systems:

$$x = x(\xi,\eta,\zeta,t) \qquad \text{(B.1a)}$$

$$y = y(\xi,\eta,\zeta,t) \qquad \text{(B.1b)}$$

$$z = z(\xi,\eta,\zeta,t) \qquad \text{(B.1c)}$$

where time is also considered in the transformation in situations where the grid structure changes with time. The goal is to express all terms of the governing hydrodynamic equations such that the independent variables are (ξ, η, ζ, t). Later we will consider various methods for ensuring conservation of physically important quantities. For the moment, let us consider a rather fundamental example in which velocity gradient terms are transformed from physical to computational space. In matrix form, we have, via the chain rule,

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} & \frac{\partial u}{\partial \zeta} \\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta} & \frac{\partial v}{\partial \zeta} \\ \frac{\partial w}{\partial \xi} & \frac{\partial w}{\partial \eta} & \frac{\partial w}{\partial \zeta} \end{bmatrix} \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{bmatrix} \quad \text{(B.2a)}$$

where the Jacobian matrix [J] of the transformation is given by

$$[J] \equiv \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} & \frac{\partial \xi}{\partial z} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} & \frac{\partial \eta}{\partial z} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} & \frac{\partial \zeta}{\partial z} \end{bmatrix} \quad \text{(B.2b)}$$

(Fletcher, 1988). That is, the Jacobian has the job of mapping the variables from one coordinate system to the other. Although one could easily work with the components of the matrix [J], the individual terms, as written, utilize the *physical coordinates* as independent variables; this is contrary to our desire of performing all computations in the transformed coordinate system where the independent variables are (ξ,η,ζ,τ). Consequently, it is more convenient to work with the inverse Jacobian matrix, given by

$$[J]^{-1} \equiv \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} . \quad \text{(B.2c)}$$

Relationships between elements of $[J]$ and those of its inverse will be given below. First, however, we note that, because the curvilinear coordinates are not Cartesian, we must distinguish between the covariant and contravariant base vectors and quantities. The **covariant base vectors** are tangent to the curvilinear coordinates, while the **contravariant** base vectors are orthogonal (TWM p. 97-98).

B.1.1. Covariant Equations

Consider a coordinate line along which only the coordinate ξ varies. A tangent vector to this coordinate line is given by

$$\lim_{d\xi \to 0} \frac{\vec{r}\,(\xi + d\xi) - \vec{r}\,(\xi)}{d\xi} = \frac{d\vec{r}}{d\xi}\,.$$

These tangent vectors to the three coordinate lines are the three **covariant base vectors** of the curvilinear coordinate system, and are given by

$$\hat{a}_i = \frac{\partial \vec{r}}{\partial \xi^i} \; (i = 1,2,3) \qquad \text{(B.3a)}$$

where the three curvilinear coordinates (ξ,η,ζ) are represented by ξ^i and the subscript i indicates the base vector corresponding to the ξ^i coordinate, *i.e.*, the tangent to the coordinate line along which only ξ^i varies ($\xi^1 = \xi$, $\xi^2 = \eta$, $\xi^3 = \zeta$). One can also write (B.2) as

$$\begin{aligned} \hat{a}_1 &= x_\xi \hat{i} + y_\xi \hat{j} + z_\xi \hat{k} \\ \hat{a}_2 &= x_\eta \hat{i} + y_\eta \hat{j} + z_\eta \hat{k} \\ \hat{a}_3 &= x_\zeta \hat{i} + y_\zeta \hat{j} + z_\zeta \hat{k}. \end{aligned} \qquad \text{(B.3b)}$$

Associated with the covariant base vectors are covariant metric components, which are used to represent differential increments of arc length, surface area, and volume (see TWM, p. 100-102). Presented here are the components of the covariant 3 x 3 symmetric metric tensor. In general,

$$g_{ij} = \hat{a}_i \cdot \hat{a}_j = g_{ji}\,. \qquad \text{(B.4)}$$

The components of this tensor are:

$$g_{11} = \hat{a}_1 \cdot \hat{a}_1 = x_\xi^2 + y_\xi^2 + z_\xi^2$$

$$g_{22} = \hat{a}_2 \cdot \hat{a}_2 = x_\eta^2 + y_\eta^2 + z_\eta^2 \qquad \text{(B.5a)}$$

$$g_{33} = \hat{a}_3 \cdot \hat{a}_3 = x_\zeta^2 + y_\zeta^2 + z_\zeta^2$$

$$g_{13} = g_{31} = \hat{a}_1 \cdot \hat{a}_3 = x_\xi x_\zeta + y_\xi y_\zeta + z_\xi z_\zeta$$

$$g_{23} = g_{21} = \hat{a}_2 \cdot \hat{a}_3 = x_\eta x_\zeta + y_\eta y_\zeta + z_\eta z_\zeta \qquad \text{(B.5b)}$$

$$g_{21} = g_{12} = \hat{a}_1 \cdot \hat{a}_2 = x_\xi x_\eta + y_\xi y_\eta + z_\xi z_\eta .$$

One can easily show by substitution that the Jacobian of the transformation is related to the determinant of the inverse Jacobian matrix (B.2c) by

$$\sqrt{G} = \sqrt{\det\left|g_{ij}\right|} = \hat{a}_1 \cdot (\hat{a}_2 \, x \, \hat{a}_3) = \left|J^{-1}\right| \qquad \text{(B.6)}$$

(TWM, p. 102, Eq. 16; Fletcher, 1988, p. 51, Eq. 12.13) which can be written in a more familiar manner as

$$\sqrt{G} \equiv \frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)} = \begin{vmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{vmatrix} . \qquad \text{(B.7)}$$

Physically, the Jacobian of transformation $\sqrt{G}$ relates contributions to the distances of arc length Δs to small changes in the computational coordinate via

$$Area = \sqrt{G}\ \Delta\xi\Delta\eta$$

where, in the two-dimensional case, $\Delta\xi$ and $\Delta\eta$ represent grid spacings in the computational coordinates. See Fletcher (1988, p. 50-51) for further details.

B.1.2. Relations Between Covariant and Contravariant Forms (TWM, p. 108, 123)

The **contravariant base vector** is normal to the coordinate surface on which the coordinate ξ is a constant and is given by

$$\hat{a}^i = \nabla \xi^i \quad (i = 1,2,3) \tag{B.8a}$$

or, when expanded, by

$$\begin{aligned} \hat{a}^1 &= \nabla \xi = \hat{i}\,\xi_x + \hat{j}\,\xi_y + \hat{k}\,\xi_z \\ \hat{a}^2 &= \nabla \eta = \hat{i}\,\eta_x + \hat{j}\,\eta_y + \hat{k}\,\eta_z \\ \hat{a}^3 &= \nabla \zeta = \hat{i}\,\zeta_x + \hat{j}\,\zeta_y + \hat{k}\,\zeta_z. \end{aligned} \tag{B.8b}$$

Note that the coordinate i appears as a superscript on the base vector in (B.8a). We will adopt the convention that superscripts indicate contravariant quantities and subscripts indicate covariant quantities.

The contravariant base vectors can be expressed as functions of the covariant base vectors using well-known expressions (TWM, p. 108-109):

$$\hat{a}^i = \frac{1}{\sqrt{G}}\, \hat{a}_j \, x \, \hat{a}_k \quad (i = 1,2,3;\ i,j,k \text{ cyclic})\ . \tag{B.9}$$

Also,

$$\hat{a}_i \cdot \hat{a}^j = \frac{1}{\sqrt{G}}\, \hat{a}_i \cdot (\hat{a}_k \, x \, \hat{a}_l) \quad [j,k,l \text{ cyclic}] \tag{B.10}$$

and thus

$$\hat{a}_i \cdot \hat{a}^j = \delta_{ij}\ . \tag{B.11}$$

Consequently, any vector $\vec{A}$ can be written in terms of either set of base vectors:

$$\vec{A} = \sum_{i=1}^{3} (\hat{a}^i \cdot \vec{A})\, \hat{a}_i \tag{B.12a}$$

$$\vec{A} = \sum_{i=1}^{3} (\hat{a}_i \cdot \vec{A})\, \hat{a}^i\ . \tag{B.12b}$$

where $A^i = \hat{a}^i \cdot \vec{A}$ is the **contravariant component** and $A_i = \hat{a}_i \cdot \vec{A}$ is the **covariant component** of the vector $\vec{A}$.

The components of the 3 x 3 symmetric **contravariant tensor** may be expressed as functions of the **covariant** components as follows:

$$g^{11} = \frac{1}{G}\left(g_{22}g_{33} - g_{23}^2\right)$$
$$g^{22} = \frac{1}{G}\left(g_{33}g_{11} - g_{13}^2\right) \qquad \text{(B.13a)}$$
$$g^{33} = \frac{1}{G}\left(g_{11}g_{22} - g_{12}^2\right)$$

$$g^{12} = g^{21} = \frac{1}{G}\left(g_{23}g_{31} - g_{21}g_{33}\right)$$
$$g^{13} = g^{31} = \frac{1}{G}\left(g_{21}g_{32} - g_{22}g_{31}\right) \qquad \text{(B.13b)}$$
$$g^{23} = g^{32} = \frac{1}{G}\left(g_{31}g_{12} - g_{32}g_{11}\right).$$

In general, we have

$$g^{il} = g^{li} = \frac{1}{G}\left(g_{jm}g_{kn} - g_{jn}g_{km}\right) \quad [i = 1,2,3 \text{ cyclic}, l = 1,2,3 \text{ cyclic}]. \qquad \text{(B.14a)}$$

We can, using expressions developed earlier, now relate the elements of the Jacobian matrix (in which *x*, *y*, and *z* are independent variables) to those of its inverse where *x*, *y*, and *z* are the desired dependent variables by noting that

$$[J] = \frac{\text{Transpose of Cofactor of}\left[J^{-1}\right]}{\left|J^{-1}\right|}$$

(Fletcher, 1988, p. 39, Eq. 12.7). This yields

$$\xi_x = \frac{(y_\eta z_\zeta - y_\zeta z_\eta)}{\sqrt{G}}, \quad \xi_y = \frac{(z_\eta x_\zeta - z_\zeta x_\eta)}{\sqrt{G}}, \quad \xi_z = \frac{(x_\eta y_\zeta - x_\zeta y_\eta)}{\sqrt{G}},$$
$$\eta_x = \frac{(y_\zeta z_\xi - y_\xi z_\zeta)}{\sqrt{G}}, \quad \eta_y = \frac{(z_\zeta x_\xi - z_\xi x_\zeta)}{\sqrt{G}}, \quad \eta_z = \frac{(x_\zeta y_\xi - x_\xi y_\zeta)}{\sqrt{G}},$$
$$\zeta_x = \frac{(y_\xi z_\eta - y_\eta z_\xi)}{\sqrt{G}}, \quad \zeta_y = \frac{(z_\xi x_\eta - z_\eta x_\xi)}{\sqrt{G}}, \quad \zeta_z = \frac{(x_\xi y_\eta - x_\eta y_\xi)}{\sqrt{G}}.$$

(B.14b)

In order to improve the conservation properties of the continuous system, we use the Cartesian velocity components rather than the contravariant components (see *e.g.*, Vinokur, 1974; Viviand, 1974; Sharman *et al.*, 1988; and Shyy and Vu, 1991), and express all of the operators in terms of contravariant variables in conservation form (see Section 7). The following identities (see TWM, p. 111) will therefore be useful in subsequent sections:

$$\nabla\phi = \frac{1}{\sqrt{G}} \sum_{i=1}^{3} \frac{\partial}{\partial\xi^i} \left(\sqrt{G}\ \hat{a}^i\ \phi\right)$$

(Conservative Gradient) (B.15a)

$$\nabla \cdot \vec{A} = \frac{1}{\sqrt{G}} \sum_{i=1}^{3} \frac{\partial}{\partial\xi^i} \left(\sqrt{G}\ \hat{a}^i \cdot \vec{A}\right)$$

(Conservative Divergence) (B.15b)

$$\nabla^2\phi = \frac{1}{\sqrt{G}} \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial}{\partial\xi^i} \left[\sqrt{G}\ g^{ij}\ \frac{\partial\phi}{\partial\xi^j}\right]$$

(Conservative Laplacian) (B.15c)

$$\left(\frac{\partial A}{\partial t}\right)_{\underset{\sim}{x}} = \left(\frac{\partial A}{\partial t}\right)_{\underset{\sim}{\xi}} - \underset{\sim}{\dot{x}} \cdot \nabla A$$

(Time Derivative Transformation) (B.15d)

where, in the latter expression, $\underset{\sim}{\dot{x}}$ is the vector speed of the moving grid (TWM, Eq. 117, p. 129).

It is also often useful to express the contravariant base vectors in terms of the covariant base vectors as follows. From Sharman *et al.* (1988), we know that

$$\begin{aligned}\nabla\xi &= \hat{a}^1 = g^{11}\,\hat{a}_1 + g^{12}\,\hat{a}_2 + g^{13}\,\hat{a}_3\\ \nabla\eta &= \hat{a}^2 = g^{21}\,\hat{a}_1 + g^{22}\,\hat{a}_2 + g^{23}\,\hat{a}_3\\ \nabla\zeta &= \hat{a}^3 = g^{31}\,\hat{a}_1 + g^{32}\,\hat{a}_2 + g^{33}\,\hat{a}_3.\end{aligned} \tag{B.16}$$

Using (B.9) and (B.13), we have the inverse transformation:

$$\begin{aligned}(\xi)_{x_i} &= \frac{1}{G}\,[\,(g_{22}g_{33} - g_{23}^2)(x_i)_\xi + (g_{23}g_{31} - g_{21}g_{33})(x_i)_\eta + (g_{12}g_{32} - g_{22}g_{31})(x_i)_\zeta\,]\\ (\eta)_{x_i} &= \frac{1}{G}\,[\,(g_{23}g_{31} - g_{21}g_{33})(x_i)_\xi + (g_{33}g_{11} - g_{13}^2)(x_i)_\eta + (g_{31}g_{12} - g_{32}g_{11})(x_i)_\zeta\,]\\ (\zeta)_{x_i} &= \frac{1}{G}\,[\,(g_{12}g_{32} - g_{22}g_{31})(x_i)_\xi + (g_{31}g_{12} - g_{32}g_{11})(x_i)_\eta + (g_{11}g_{22} - g_{12}^2)(x_i)_\zeta\,]\end{aligned} \tag{B.17}$$

where the subscript i indicates the direction of differentiation, *e.g.*, $x_2 = y$, $\xi_3 = \zeta$, *etc.*

B.2. Application of the Transformation Relations to a Boussinesq System of Conservation Laws

It is useful to illustrate the application of the transformation relations in Section 6 to a somewhat simpler system (*e.g.*, Sharman *et al.*, 1988). The generalization to fully compressible equations is straightforward.

B.2.1. Scalar Transport Equation

Consider first the conservation equation for a scalar ϕ in an incompressible fluid:

$$\frac{\partial\phi}{\partial t} + \nabla\cdot(\vec{V}\phi) = S \tag{B.18}$$

where $\vec{V}$ is the velocity vector and S represents possible sources and sinks. Applying (B.15b) to (B.18) and noting that the **contravariant velocity components** are given by

$$U^c = \vec{V} \cdot \hat{a}^1$$
$$V^c = \vec{V} \cdot \hat{a}^2 \tag{B.19}$$
$$W^c = \vec{V} \cdot \hat{a}^3$$

we have, *for the case in which the coordinate transformation is invariant with time* (Sharman *et al.*, 1988, p. 1126, Eq. 7b),

$$\frac{\partial(\sqrt{G}\ \phi)}{\partial t} + \frac{\partial(\sqrt{G}\ U^c\ \phi)}{\partial \xi} + \frac{\partial(\sqrt{G}\ V^c\ \phi)}{\partial \eta} + \frac{\partial(\sqrt{G}\ W^c\ \phi)}{\partial \zeta} = S\sqrt{G}\ . \tag{B.20}$$

In this case, the quantity $\sqrt{G}\ \phi$ is conserved, and this equation is said to be in *strong conservation law form* (for exhaustive treatments of various conservation law forms, see Thomas and Lombard, 1978; Hindman, 1981; Anderson *et al.*, 1984).

B.2.2. Momentum Equation

In the case of momentum, Sharman *et al.* (1988) and, more recently, Shyy and Vu (1991) discuss the difficulties encountered when writing the conservation form. Basically, we wish to have the equations in the transformed system obey the same conservation properties as those in the physical system. To illustrate the issue, consider the momentum equation for a Boussinesq fluid:

$$\frac{\partial\vec{V}}{\partial t} + \nabla\cdot(\vec{V}\vec{V}) = -\nabla\left(\frac{p'}{\rho_o}\right) + B\hat{k} \tag{B.21}$$

where p' is the deviation of pressure from hydrostatic balance, ρ_o is the reference density, and B represents the buoyancy. The term of concern in (B.21) is the nonlinear flux, which is given by the divergence of a dyadic:

$$\nabla\cdot(\vec{V}\vec{V}). \tag{B.22a}$$

where $\vec{V}$ is the velocity.

Using rules similar to those defined in Section 6 for the divergence of a vector, we can expand (B.22a) to yield (Sharman *et al.*, 1988)

$$\nabla \cdot (\vec{V}\vec{V}) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial \xi} [\sqrt{G}\, \vec{V}(\vec{V} \cdot \hat{a}^1)] + \frac{1}{\sqrt{G}} \frac{\partial}{\partial \eta} [\sqrt{G}\, \vec{V}(\vec{V} \cdot \hat{a}^2)] + \frac{1}{\sqrt{G}} \frac{\partial}{\partial \zeta} [\sqrt{G}\, \vec{V}(\vec{V} \cdot \hat{a}^3)]. \tag{B.22b}$$

But, we also know from (B.12a) and (B.19) that

$$\vec{V} = U^c\, \hat{a}_1 + V^c\, \hat{a}_2 + W^c\, \hat{a}_3 \tag{B.23}$$

so that the momentum equation in vector form becomes (Sharman *et al.*, 1988; Eq. 9):

$$\frac{\partial}{\partial t} [\sqrt{G}(U^c \hat{a}_1 + V^c \hat{a}_2 + W^c \hat{a}_3)] + \frac{1}{\sqrt{G}} \frac{\partial}{\partial \xi} [\sqrt{G}\, U^c(U^c \hat{a}_1 + V^c \hat{a}_2 + W^c \hat{a}_3)] + \frac{1}{\sqrt{G}} \frac{\partial}{\partial \eta} [\sqrt{G}\, V^c(U^c \hat{a}_1 + V^c \hat{a}_2 + W^c \hat{a}_3)] + \frac{1}{\sqrt{G}} \frac{\partial}{\partial \zeta} [\sqrt{G}\, W^c(U^c \hat{a}_1 + V^c \hat{a}_2 + W^c \hat{a}_3)] = \left[- \nabla \frac{p'}{\rho_o} + B\hat{k} \right] \sqrt{G}. \tag{B.24}$$

(Note that B.23 may be written equivalently as $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$; to show this, simply use the definitions of U^c, V^c, and W^c, Eq. (B.23), and the contravariant base vectors — basically, this exercise verifies Eqs. (B.12).)

As clearly pointed out by Sharman *et al.* (1988), two decompositions or choices of velocity variables are possible from this point, the choice being somewhat problematic (scalars don’t enter here because they have no directional dependence). First of all, it is impossible to obtain a fully conservative form of the hydrodynamic equations when using either the covariant or contravariant velocities (*e.g.*, Shyy and Vu, 1991) since linear momentum is conserved along a straight line, not a curved one (note that in a Cartesian system, this problem does not exist). Consequently, if the governing equations are written using either the covariant or contravariant velocities, spurious source terms will arise due to the curvature of the coordinate system. On the other hand, the covariant and contravariant velocities are defined in terms of local coordinates, not fixed coordinates as for Cartesian velocity components, thus making the former preferable to the latter if conservation isn’t a major issue. Furthermore, as pointed out by Sharman *et al.* (1988) from earlier work by Vinokur (1974) and Viviand (1974) a decomposition of the governing equations along the curvilinear coordinates would involve a differentiation of the base vectors. Though straightforward, this procedure is deemed inadvisable since the base vectors are already determined by the derivatives of the position vector $\vec{r}$, and a further differentiation may increase the vulnerability of the solution to truncation error.

After considering these and other issues, we chose for illustration purposes to follow the strategy of most transformed models and use the Cartesian velocity components in the transformed coordinate system. This methodology decouples the velocity components from coordinate variations, and is advantageous since most scientists can readily identify with Cartesian velocity components. Since Sharman *et al*. (1988) used this approach for studying orographic flows on orthogonal and nonorthogonal grids, we will basically follow their treatment since it is succinct and readily available in the meteorological literature.

First, we employ the chain rule to write the Cartesian velocity components as functions of the contravariant velocity components (assuming for the moment that the grid does not change in time):

$$\begin{aligned} u &= U^c \frac{\partial x}{\partial \xi} + V^c \frac{\partial x}{\partial \eta} + W^c \frac{\partial x}{\partial \zeta} \\ v &= U^c \frac{\partial y}{\partial \xi} + V^c \frac{\partial y}{\partial \eta} + W^c \frac{\partial y}{\partial \zeta} \\ v &= U^c \frac{\partial z}{\partial \xi} + V^c \frac{\partial z}{\partial \eta} + W^c \frac{\partial z}{\partial \zeta}. \end{aligned} \tag{B.25}$$

The inverse transformation is given by

$$\begin{aligned} U^c &= u\xi_x + v\xi_y + w\xi_z \\ V^c &= u\eta_x + v\eta_y + w\eta_z \\ W^c &= u\zeta_x + v\zeta_y + w\zeta_z \end{aligned} \tag{B.26a}$$

where, once again, U^c, V^c, and W^c are the contravariant velocity components. As mentioned earlier, we desire that the independent variables be (ξ, η, ζ) rather than (x, y, z), and thus we can use (B.14b) to express (B.26a) more meaningfully as

$$\begin{aligned} U^c\sqrt{G} &= uJ^{yz}_{\eta\zeta} + vJ^{zx}_{\eta\zeta} + wJ^{xy}_{\eta\zeta} \\ V^c\sqrt{G} &= uJ^{yz}_{\zeta\xi} + vJ^{zx}_{\zeta\xi} + wJ^{xy}_{\zeta\xi} \\ W^c\sqrt{G} &= uJ^{yz}_{\xi\eta} + vJ^{zx}_{\xi\eta} + wJ^{xy}_{\xi\eta} \end{aligned} \tag{B.26b}$$

where the Jacobians of transformation are:

$$
\begin{aligned}
&J^{yz}_{\eta\zeta} \equiv \frac{\partial(y,z)}{\partial(\eta,\zeta)} &\quad J^{zx}_{\eta\zeta} \equiv \frac{\partial(z,x)}{\partial(\eta,\zeta)} &\quad J^{xy}_{\eta\zeta} \equiv \frac{\partial(x,y)}{\partial(\eta,\zeta)} \\
&J^{yz}_{\zeta\xi} \equiv \frac{\partial(y,z)}{\partial(\zeta,\xi)} &\quad J^{zx}_{\zeta\xi} \equiv \frac{\partial(z,x)}{\partial(\zeta,\xi)} &\quad J^{xy}_{\zeta\xi} \equiv \frac{\partial(x,y)}{\partial(\zeta,\xi)} \\
&J^{yz}_{\xi\eta} \equiv \frac{\partial(y,z)}{\partial(\xi,\eta)} &\quad J^{zx}_{\xi\eta} \equiv \frac{\partial(z,x)}{\partial(\xi,\eta)} &\quad J^{xy}_{\xi\eta} \equiv \frac{\partial(x,y)}{\partial(\xi,\eta)}.
\end{aligned}
\tag{B.27}
$$

In this manner, we have effectively redefined the inverse transformation relations as follows:

$$
\begin{aligned}
&J^{yz}_{\eta\zeta} = \sqrt{G}\ \xi_x &\quad J^{zx}_{\eta\zeta} = \sqrt{G}\ \xi_y &\quad J^{xy}_{\eta\zeta} = \sqrt{G}\ \xi_z \\
&J^{yz}_{\zeta\xi} = \sqrt{G}\ \eta_x &\quad J^{zx}_{\zeta\xi} = \sqrt{G}\ \eta_y &\quad J^{xy}_{\zeta\xi} = \sqrt{G}\ \eta_z \\
&J^{yz}_{\xi\eta} = \sqrt{G}\ \zeta_x &\quad J^{zx}_{\xi\eta} = \sqrt{G}\ \zeta_y &\quad J^{xy}_{\xi\eta} = \sqrt{G}\ \zeta_z.
\end{aligned}
\tag{B.28}
$$

Eq. (B.26b) is also more appropriate for use in the governing equations than (B.26a) since the former includes terms of the form $U^c\sqrt{G}$, matching those in the flux terms of the momentum equation (B.24). (It is immediately clear from B.26a that the velocities in the transformed system are interdependent. Consequently, one must put aside traditional thinking to avoid the notion that an x-derivative of the zonal momentum involves only differences in the x-direction; in the transformed system, derivatives in one direction contain terms in the other two coordinate directions as well!)

Using (B.26b) and the fact that $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$, we can write the terms to be differentiated in the momentum equations as

$$
\begin{aligned}
\sqrt{G}\ U^c\,\vec{V} &= u\sqrt{G}\ U^c\,\hat{i} + v\sqrt{G}\ U^c\,\hat{j} + w\sqrt{G}\ U^c\,\hat{k} \\
\sqrt{G}\ V^c\,\vec{V} &= u\sqrt{G}\ V^c\,\hat{i} + v\sqrt{G}\ V^c\,\hat{j} + w\sqrt{G}\ V^c\,\hat{k} \\
\sqrt{G}\ W^c\,\vec{V} &= u\sqrt{G}\ W^c\,\hat{i} + v\sqrt{G}\ W^c\,\hat{j} + w\sqrt{G}\ W^c\,\hat{k}.
\end{aligned}
\tag{B.29}
$$

This allows us to decompose the momentum equations along the (i, j, k) directions using the Cartesian velocity component in the transformed, curvilinear system.

Before doing this, let us return to the continuity equation (B.20) and express it using the Cartesian velocity components (B.26b):

$$\frac{\partial(\sqrt{G}\ \phi)}{\partial t} + \frac{\partial\,[\phi\,(uJ^{yz}_{\eta\zeta} + vJ^{zx}_{\eta\zeta} + wJ^{xy}_{\eta\zeta})]}{\partial\xi}$$
$$+ \frac{\partial[\phi\,(uJ^{yz}_{\zeta\xi} + vJ^{zx}_{\zeta\xi} + wJ^{xy}_{\zeta\xi})}{\partial\eta} \qquad \text{(B.30)}$$
$$+ \frac{\partial[\phi\,(uJ^{yz}_{\xi\eta} + vJ^{zx}_{\xi\eta} + wJ^{xy}_{\xi\eta})}{\partial\zeta} = S\sqrt{G}\,.$$

This equation is still written in strong conservation law form, and will conserve the property $\sqrt{G}\ \phi$ if an appropriate discretization is used for the spatial derivatives. In order to reduce the visual complexity of this equation, one could solve it using the form given by (B.20) and simply define the contravariant velocities (B.26b) as separate quantities to be pre-computed each time-step and inserted during the integration.

Returning finally to the momentum equation (B.24), we substitute $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$ in the time derivative and use (B.29) in the flux terms to arrive at the final form that is readily decomposed along the $\hat{i}, \hat{j}, \hat{k}$ directions utilizing Cartesian velocity components in the transformed coordinate system (note that we have used B.15a to obtain the pressure gradient terms):

Zonal momentum equation:

$$\frac{\partial}{\partial t}(u\sqrt{G}) =$$

$$-\ [\frac{\partial}{\partial\xi}\,\{u\,(uJ^{yz}_{\eta\zeta} + vJ^{zx}_{\eta\zeta} + wJ^{xy}_{\eta\zeta})\} + \frac{\partial}{\partial\eta}\,\{u\,(uJ^{yz}_{\zeta\xi} + vJ^{zx}_{\zeta\xi} + wJ^{xy}_{\zeta\xi})\}$$
$$+ \frac{\partial}{\partial\zeta}\,\{u\,(uJ^{yz}_{\xi\eta} + vJ^{zx}_{\xi\eta} + wJ^{xy}_{\xi\eta})\}\ ]$$

$$-\ [\frac{\partial}{\partial\xi}\,(\frac{P'}{\rho_o}\,J^{yz}_{\eta\zeta}) + \frac{\partial}{\partial\eta}\,(\frac{P'}{\rho_o}\,J^{yz}_{\zeta\xi}) + \frac{\partial}{\partial\zeta}\,(\frac{P'}{\rho_o}\,J^{yz}_{\xi\eta})].$$

(B.31a)

Meridional momentum equation:

$$\frac{\partial}{\partial t}(v\sqrt{G}) =$$

$$- \left[\frac{\partial}{\partial \xi} \{v\,(uJ^{yz}_{\eta\zeta} + vJ^{zx}_{\eta\zeta} + wJ^{xy}_{\eta\zeta})\} + \frac{\partial}{\partial \eta} \{v\,(uJ^{yz}_{\zeta\xi} + vJ^{zx}_{\zeta\xi} + wJ^{xy}_{\zeta\xi})\}\right.$$
$$\left. + \frac{\partial}{\partial \zeta} \{v\,(uJ^{yz}_{\xi\eta} + vJ^{zx}_{\xi\eta} + wJ^{xy}_{\xi\eta})\}\right]$$

$$- \left[\frac{\partial}{\partial \xi}(\frac{P'}{\rho_o} J^{zx}_{\eta\zeta}) + \frac{\partial}{\partial \eta}(\frac{P'}{\rho_o} J^{zx}_{\zeta\xi}) + \frac{\partial}{\partial \zeta}(\frac{P'}{\rho_o} J^{zx}_{\xi\eta})\right].$$

(B.31b)

Vertical momentum equation:

$$\frac{\partial}{\partial t}(w\sqrt{G}) =$$

$$- \left[\frac{\partial}{\partial \xi} \{w\,(uJ^{yz}_{\eta\zeta} + vJ^{zx}_{\eta\zeta} + wJ^{xy}_{\eta\zeta})\} + \frac{\partial}{\partial \eta} \{w\,(uJ^{yz}_{\zeta\xi} + vJ^{zx}_{\zeta\xi} + wJ^{xy}_{\zeta\xi})\}\right.$$
$$\left. + \frac{\partial}{\partial \zeta} \{w\,(uJ^{yz}_{\xi\eta} + vJ^{zx}_{\xi\eta} + wJ^{xy}_{\xi\eta})\}\right]$$

$$- \left[\frac{\partial}{\partial \xi}(\frac{P'}{\rho_o} J^{xy}_{\eta\zeta}) + \frac{\partial}{\partial \eta}(\frac{P'}{\rho_o} J^{xy}_{\zeta\xi}) + \frac{\partial}{\partial \zeta}(\frac{P'}{\rho_o} J^{xy}_{\xi\eta})\right] - B\sqrt{G}.$$

(B.31c)